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Culture is one of the key factors playing a role in learning and teaching mathematical proof, possibly not as much as learners understanding of what proving means out of school life but quite—there is on the latter at least all the work on the relationships between natural and mathematical logic. So, if we consider that culture has a role to play in understanding what becomes a mathematical proof from a teaching- learning perspective, we should have to raise the question of its impact on research itself.

Is there a shared meaning of "mathematical proof" among researchers in mathematics education?[1] I have every expectation that almost all researchers will agree on a more or less formal definition of mathematical proof. But beyond that what is the state of our field? To get a first insight for a possible answer, I went through a large number of research papers to figure out whether beyond the keywords we had some common understanding[2]. To discover that it is not the case was in fact not surprising. The issue then is to explore where are the differences and what is the price for them in our research economy. My main concern is that if we do not clarify this point, it will be hardly possible to share results and hence to make any real progress in the field. I do not expect every researcher to come on a same line, but we may benefit from being able to witness our convergences and to turn our differences into research questions.

Before going ahead, I would like to emphasize a fundamental remark: *rationality is dense everywhere in human being life*, either at an individual or a collective level. By "rationality" is meant here the system of the criteria or rules mobilized when one has to make choices, to take decisions, or to perform judgements. Indeed, a large part of our life has to do with informing, claiming, discussing, arguing. These rules and criteria could either be taken for granted and remain implicit—what is in general the case in the so-called every-day life, or they could be explicit or even formalised—what is the case when one has to justify his or her claim for the truth or the validity of a statement or an action. These rules and criteria could originate in opinion, belief or knowing[3] but in all cases they are organised in a structure, which allows decision-making. Searle (2001) notices that between knowing and deciding, as well as between deciding and acting, there is a gap and he proposes that it is rationality that allows human beings—as individuals or collectively—to fill in this gap. This is a very large understanding of rationality, but it holds the advantage of witnessing the complexity to which we are confronted.

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[1] I first raised this question in the Proof Newsletter (September-October 1999 issue). See in particular the Sekiguchi contribution in the January-February 2000 issue.
[2] This has been the key point of the lecture I gave to the 1995 summer school in Didactique of mathematics
[3] In this text, as we did in the translation of Brousseau work, I will use "knowing" as a noun to convey the useful distinction we have in French and most roman language between "connaissance" and "savoir"—the later being translated by "knowledge" in English.

Rationality allows us to reason and to decide, it is then the foundation of any proving process. How we see rationality in general, and its relation to mathematics in particular, is a key point for our understanding of any piece of work in our field of research. One may accept, with not too much difficulty, that proving depends on content and context. Indeed, the issue of truth or validity cannot be settled in the same way in everyday life, in politics, in philosophy, in medicine, in physics or in mathematics. And it is clear that one does not mobilize the same rules and criteria for decision-making depending of the context in which he or she is involved—what we sometimes refer to by the expression "economy of logic".

How do we position ourselves with respect to these aspects? How do we take into account the tangle of context and content? The way we answer these questions is key in determining our view of what a mathematical proof is from a teaching-learning point of view. The way we choose to answer these questions, either consciously or not, speaks for our epistemology of proof and for our own rationality.

Our epistemology of proof—in short, the acquaintance we have with truth and validity—first shapes our research framework, even before the choice of a *problématique* (the choice of the relevant questions and research problems), and the choice of a theoretical framework and its related methodology. I do not see that this issue has been addressed, in my best knowledge, although to be aware of it and to explore it systematically may be what conditions both the quality of what we produce and the possibility to exchange results. My claim is that our epistemology of proof is the first deadlock to figure out and to cope with, when entering our research field. One may easily agree that this is especially crucial for young researchers, who generally enter the field with a naïve or intuitive *problématique*. My claim is that this is a deadlock for the whole field. Unless we have clarified precisely what this deadlock is like and how it limits our capacity to share research outcomes, it will be hardly possible to make significant progress in the field.

This talk will be organised around snapshots at some piece of research I consider as being representative of the main research trends in our field. The way I will sketch them is a bit risky, since there is much more in these works than what I kept for the purpose of my talk. Fortunately, several of the authors I quote, participate in this conference, so they will be able to react and correct me if necessary. The presentation of the snapshots will be organised so as to insist on the contrasts between the different approaches. In any case, this talk intends to open a collective reflection on our research, it will end on a call for opening a workshop on the impact of researchers epistemology on their own work. That is, a call for a *Taipei manifesto for educational research on mathematical proof*.

1

> "The concept of proof is one concerning which the pupil should have a growing and increasing understanding. It is a concept which not only pervades his work in mathematics but is also involved in all situations where conclusions are to be reached and decision to be made. Mathematics has the unique contribution to make in the development of this concept." (Fawcet, 1938)

Mathematics, by essence the science shaped by logic, could be viewed as a reference to proving in general, and beyond as the best example of rationality. This understanding of mathematics has been widely shared, and in France too we could have adopted this position—mathematics being the place for the education of deductive reasoning.

This seems to have been systematised in a rather radical way by the US mathematics educators of the first part of the last century. In his seminal book, *"The nature of proof"*, Fawcet demonstrates how considering any claim, as for example the one presented here after, under the light of arepresentation format (a three columns table) favouring an analytical approach, may prepare students to understand the logical power of mathematics as a science of formal reasoning.

3. Racial superiority.
Many people believe that the white races are superior to the coloured races.
My present belief concerning this is:

Analysis of my belief	Statement of the facts	Assumption

The idea is that the validity of a statement—either an opinion, or a belief, or a knowing[4]—could be scrutinized with the help of such a format, which scaffolds the elicitation of its rational. For mathematics education, the didactical object in line with this approach is the well-known two columns proof which facilitates the evaluation of a proof by both the student and the teacher (see Herbst work on the didactical transposition of mathematical proof in the US).

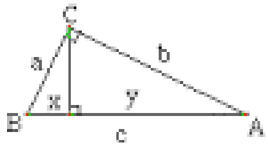
Theorem 7.4 Pythagorean Theorem

In a right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the legs.

Given: △ABC is a right triangle.

∠C is a right angle.

Prove: $a^2 + b^2 = c^2$



Proof:

Statements	Reasons
1. △ABC is a right triangle. ∠C is a right angle.	1. Given
2. Draw a perpendicular from C to AB.	2. Theorem 2.9
3. c/a=a/x , c/b=b/y	3. Theorem 7.3
4. a^2=cx , b^2=cy	4. Product of mean equal product of extremes.
5. a^2+b^2=cx+cy	5. Addition Property
6. a^2+b^2=c(x+y)	6. Distributive Property
7. a^2+b^2=c2	7. Segment Addition Postulate, Substitution

Indeed, this process of *didactification*, as it is very often the case—and as it is widely known in the case of proof—, "killed" he initial intention Fawcett had. But I will leave to US researchers the responsibility to develop further this judgement:

[4] Following what we did for the translation of Brousseau' theory of didactical situations, I will here use the word "knowing" to keep track of the difference we have in roman languages between *connaissance* and *savoir*; the later being translated by the word "knowledge". A "knowing" is a personal construct, which becomes a "knowledge" if it is shared by a community under an institutionalised form.

> Herbst analyses that "What was at stake was not just to teach for transfer but also to show that such transfer could be demonstrated. […] The students' logical reasoning had to be measurable according to the standards of the measurement movement, and the two-column proof format could be adapted to furnish measurement instruments" (Herbst 1999).

The emphasis on the formal dimension of mathematical proof, reified by the two-column proof format, led to strong reaction in the last part of the last century. As Harel and Sowder point: "students do not learn that proof are first and foremost convincing arguments, that proofs (and theorems) are a product of human activity, in which they can and should participate […] the goal is to help students define their own conception of what constitutes justification in mathematics" (Harel & Sowder 1998, p. 297). The radical reduction of mathematical proof to its formal organisation had a price that research soon pointed, which is a loss of meaning of proving or at least a shift of meaning—which indeed was not intended by Fawcett and his followers. This criticism favoured the raise of an awareness of the human dimension of any mathematical activity, and of proving in particular. It is on this line that Usiskin suggested that teaching of proof had failed "because we too often ignore: when and why mathematicians do proof, the variety of possible types of proof, and how mathematicians write down proofs." (Harel and Sowder 1998, p.419, quoting Usiskin).

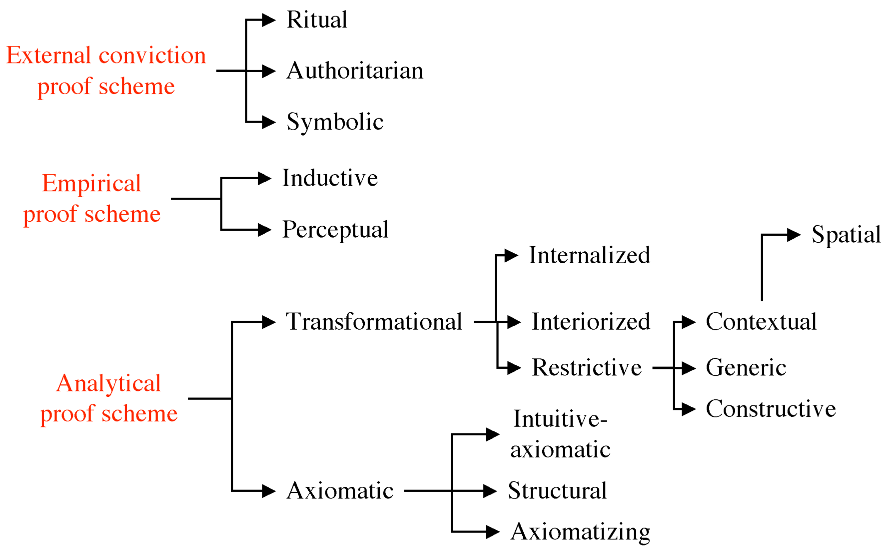
2

> "A person's proof scheme consists of what constitutes ascertaining and persuading for that person […] As defined, ascertaining and persuading are entirely subjective and can vary from person to person, civilisation to civilisation, and generation to generation within the same civilisation" (Harel & Sowder 1998, p.242). And finally: "one's proof scheme is idiosyncratic and may vary from field to field, and even within mathematics itself" (ibid. p.275).

Indeed it is in a rather drastic way, that Harel and Sowder put back the person at the centre of a *problématique* of proof. In their work, this precise characterisation of proof scheme is based on careful definitions, which in fact have the effect of reinforcing the central place of the individual as such (ibid. p.241):

- A *conjecture* is an observation made by a person who has no doubt about its truth. A person's observation ceases to be a conjecture and becomes a fact in his or her view once the person becomes certain of its truth.
- By *proving* we mean the process employed by an individual to remove or create doubts about the proof of an observation
- *Ascertaining* is the process an individual employs to remove his or her own doubt about the truth of an observation
- *Persuading* is the process an individual employs to remove other doubts about the truth of an observation

In a way, understanding proof is viewed on a continuum—which holds a genetic stance—from the more "idiosyncratic" to the more "objective", which is the more specific to mathematic as a content which transcends human beings own knowings. Harel and Sowder, after others, suggest a classification that gives account of this view, ranging from "ritual" and "authoritarian" to "structural" and "axiomatic"—the following schema summarizes it:



It must be reminded that this classification is based on an empirical study. Its strength lies in the fact that it is not abstract, nor a priori. As stated by these authors: "all [results] were derived from our observation of the actions taken by actual students in their process of proving" (ibid. p.244). The word "actual" in this quote is significant, it refers to a naturalistic approach; an approach free from the artificiality of an experimental setting. Also, Harel and Sowder insist that there is no normative hierarchy in the proposed classification. It is what it is: it is what observation delivers.

Nevertheless, the focus on the person does not mean that the content is forgotten. It appears here and there, like in the analysis of Duane's conception of a definition of a line, which points Duane's inability to represent the geometric properties stated, in any context but his own imaginary space (ibid. pp.269-270). The possible specificity of mathematics is in particular taken into account when defining the axiomatic proof scheme: "when a person understands that at least in principle a mathematical justification must have started originally from undefined terms and axioms, we say that this person possesses an axiomatic proof scheme" (ibid. p.273).

Interactions in the classroom, viewed through the lenses of this approach, encompass all the complexity of the confrontation of individuals—and a priori acceptable—proof schemes in the teaching of mathematics. The two following excerpts witness it:

> On the student side: "Bob expresses his dissatisfaction with the instructor's decision by saying that he does not understand what difference it makes if he was told the proof or if he found it on his own; the result in both cases is the same: he would know the proof" (ibid. pp.247-248).
>
> On the teacher side: "during instruction, empirical justification serve as examples of arguments given by mathematicians, and may inadvertently sanction the empirical proof scheme as a mode of justification fully acceptable in the mathematical context" (ibid. p.278).

In this context, the notion of "theorem", although it is specific to mathematics (insofar as this word is a distinctive word of the mathematical jargon), gets a psychological meaning when claiming that it is for a mathematician a statement (i) understood and nothing suggests that it is not true, significant enough to have implications in various domains and hence justifying a detailed study, (ii) the author has an outstanding reputation in the domain of the theorem, it exists a convincing argument, rigorous or not, of a known type.

3

> "Proof is the heart of mathematical thinking, and deductive reasoning, which underpins the process of proving, exemplifies the distinction between mathematics and the empirical sciences" (Healey and Hoyles 1998).

The London group recently carried out one of the largest and more comprehensive study in order to clarify students understanding and view of mathematical proof. They made this study with in the background a strong claim about the nature of proof in mathematics, which I quoted just above. This view of mathematical proof is primarily related to the technical understanding of what it is in the mathematical activity:

> "The process of building a valid proof is clearly a complex one: it involves sorting out what is given—the mathematical properties that are already known or can be assumed—from what is to be deduced, and then organising the transformations necessary to infer the second set of properties from the first into a coherent and complete sequence." (ibid.)

In the context of the UK National Curriculum for mathematics, they carried out an investigation focussing on the following items:

- To describe the characteristics of mathematical justification and proof recognised by high-attaining Year 10 students;
- To analyse how students construct proofs;
- To investigate the reasons behind students' judgements of proofs, their performance in proof construction and their methods of constructing proofs.

This study started in 1995. It involved good 10 years students and their teachers. Students were asked to construct proofs or to judge proofs. Using questionnaires and interviews in a sophisticated manner that I will not account here in details[5]. They obtained a lot of detailed results, which altogether demonstrate rather low levels of achievement in the construction of proofs, with better success in algebra than in geometry. But the same analysis demonstrates that most students understand the generality of a valid proof: students are better at recognising a valid argument than at constructing it, and their conception of proof and of its role is key in their performance. These performances appeared independent of the teacher characteristics but highly related to the number of hours devoted to mathematics, the explicit emphasis on proof and the level of students (or their familiarity with the content at stake).

One of the general results emphasised by the authors of the London group stresses the need for a more explicit teaching of mathematical proof:

> "The research indicates that the ability to construct, assess or choose a valid proof is not simply a matter of general mathematical attainment. Clearly this has an influence, but at least some of the poor performance in proof of our highest-attaining students may simply be explained by their lack of familiarity with the process of proving. Far

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[5] 182 students in the pilote study, 2459 students queries form 94 classes in 90 schools.

> too many students have little idea of this process and no sense of proof, which, our findings suggest, can hinder their ability to construct and correctly evaluate proofs." (ibid.)

The education to mathematical proof must not lead to an emphasis on the form, but on the meaning of proof within the mathematical activity. These researchers invite the community "to engage students with proof while discussing with them the idea of proof at a meta-level, in terms of its meaning, generality and purposes" (ibid.) Rigorous mathematical proofs and informal argumentations are seen as possibly cohabitating, provided that they are considered within the common framework of a larger reflection on proving and its place and role in mathematics. This research and the conclusions drawn by the London group, to some extend support David Tall suggestion:

> "The cognitive development of students needs to be taken into account so that proof are presented in forms that are potentially meaningful to them. This requires educators and mathematicians to rethink the nature of mathematical proof and to consider the use of the different types of proof related to the cognitive development of the individual" (Tall 1998).

4 "The most significant potential contribution of proof to mathematics education is the communication of mathematical understanding" […] "A mathematics curriculum which aims to reflect the real role of rigorous proof in mathematics must present it as an indispensable tool of mathematics rather than at the very core of that science" (Hanna and Janke 1996, pp.877-879)

Hanna and Janke view of mathematical proof is rather instrumental, insisting on the fact that "in the first place, formal proof arose as a response to a persistent concern for justification […] Formal mathematical proof has been and remains one quite useful answer to this concern for justification" (ibid. p.888). The position they adopt is in a clear rejection of a common fallibilist ideology of those who "appear to see proof in general, and rigorous proof in particular to help impose upon students a body of knowledge that it regards as predetermined and infallible" (ibid. p.890). They argue against a naïve view of rigour and certainty in mathematics, they support their argument by coming back to errors in the history of mathematics (ibid. p.891) and the current revision of proving standard required by the use of computers in mathematics (ibid. p.881).

In reaction to the movement against mathematical proof—the so-called formal proof—they claim that "the use of proof in the classroom is actually anti-authoritarian" (ibid. p. 891), and even that "it would be disturbing to see mathematics teachers ranging themselves on the side of a revolt against rationality" (ibid.). Hanna and Janke make then a pragmatic choice, expressed in the form of two hypotheses:

- "Hypothesis 1: communication in scholarly mathematics serves mainly to cope with mathematical complexity, while communication at schools serves more to cope with epistemological complexity"

- "Hypothesis 2: in order to understand the meaning of a theorem and the value of its proof, students must have an extensive and coherent experience in the appropriate application area. This pragmatic foundation can and should be taught in conscious separation from the formal derivation. Only then the students be able to see the real point of a proof"

(Hanna and Janke 1993, pp. 433-4)

Although Hanna and Janke do not state anywhere that they answer Tall's 1998 demand, one can observe that their hypothesis are a first possible response, and that they are stated at an epistemological level. It is not the form of mathematical proof that they put under question, valuing any other format of proof instead of the mathematical one. They advocate the search for another relation between mathematical proof and mathematics as content. I see the epistemological complexity they point as the complexity raised by the specific nature of mathematical objects, but still the way proposed to address this complexity is to bypass it by constructing a systematic link between mathematics and its application fields. Eventually, one of the distinguished feature of this approach of mathematical proof is the claim that…

> "In particular proof cannot be taught or learned without taking into consideration the relationships of mathematics to reality" (Hanna and Janke 1996, 902)

5

> "A geometrical fact, a theorem […] is acceptable only because it is systematised within a theory, with a complete autonomy from any verification or argumentation at an empirical level" (Mariotti 1997 sec 1.3)

Mariotti claim contrasts in an important manner with Hanna and Janke position, since it pretends not to search the roots of the meaning of mathematical proof primarily outside of mathematics[6]. Her position is clearly based on the recognition of a specific characteristic of mathematics: "the theoretical organisation according to axioms, definitions and theorems, represents one of the basic elements characterising mathematical knowledge" (Mariotti 1997 sec 1.4).

Mariotti research programme must be situated within the general framework of the Italian approach to teaching and learning mathematical proof, which is based on two more general concepts: *field of experience* (a concept proposed by Paolo Boero—Boero *et al.* 1995) and *mathematical discussion* (a concept proposed by Bartolini-Bussi 1996).

- "Field of experience" ensures the presence of concrete and semantically pregnant referents for performing concrete actions that allow the internalisation of the visual field where dynamic mental experiment are carried out, presence of semiotic mediation tools, construction of an evolving student internal context (conjecturing, arguing, proving take sense there)

- "Mathematical discussion" refers to a polyphony of articulated voices on a mathematical object. Mathematical discussion works as a lever to address two major issues: the need for proof, the distinction between argumentation and proof.

These concepts are related to the will to take into account two problems: (i) the relation to content since proving is always stating the validity or the truth of a statement which has a

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[6] I must immediately add that it does mean that Mariotti reduce mathematics to a formal game, on the contrary she relates it all along her work to problem solving likely to provide a "concrete meaning" to students activity. Her *problématique* is that of allowing students passing from a pragmatic to a theoretical conception of proving—hence proving getting its meaning from within mathematics.

content, (ii) the relation to language which is due to the dialogic nature of proof production (even if the person considers him or herself has his or her own interlocutor). We must be aware of the Italian group claim of the need for the existence of a reference to proving as *a system of shared principles and deduction rules*. In other words the basic equation shaping the Italian group *problématique* for research on teaching and learning mathematical proof is:

theorem=system (statement, proof, theory).

Hence, the education problem is to help students to pass from the need of *justifying* towards the idea of *validating* within a mathematical system[7] and that the "acceptance of validation depends on the meaning of the rules and on the acceptance of the rules" (Mariotti 1997 sec 4.4.).

In this framework, the Italian group search for the conditions that could allow students to access to the meaning of theorem in mathematics (in fact, precisely, in geometry) and to study the mental processes at stake. The observations, in experimental classrooms lasted a long period and considered the whole school year at different levels (5, 8 and 10). The context is given by the study of *shadows* (e.g. to study the parallelism of the shadows of two non parallel sticks). This context is chosen because shadows are "meaningful from the space geometry point of view; not easy to prove; and lacking the possibility of substituting proof with the creation of drawings" (ibid.). The teaching-learning setting covers several phases: give a problem, produce a conjecture, discuss the conjecture, work out its formulation and prepare the proof. The role of teacher is seen as being essential and is emphasised in this approach: "The teacher is responsible for introducing pupils to a theoretical perspective which is needed for a systematic view of mathematical theorems" (Mariotti *et al.* 1997 sec 3)

From this experimental study, the results drawn include the following: "[Most of the students seems] aware of the fact that they had to get the truth of the statement by reasoning" (Mariotti *et al.* 1997 sec 5) and that "their reasoning started from properties considered as true and got the truth of the statement in the 'scenario' determined by the hypothesis" (ibid.). A statement in the research report must draws our attention to the fact that "the whole activity performed by students in all the experiments shared many aspects with mathematicians work when they produce conjectures and proofs in some mathematics fields" (ibid.). Not only is mathematical proof—as an object—the reference, but also the mathematician activity recalling that proving is a process.

A more comprehensive view of the *problématique* of this approach is conveyed by the authors' claim that "classroom culture is strongly determined by recourse to mathematical discussion orchestrated by the teacher to change the spontaneous attitude of students towards theoretical validation" (ibid.). For further discussions, I would personally like to emphasise in this quotation the use of the expression "*theoretical validation*", which in my own understanding capture the essence of the Italian group approach.

[7] Actually, I am here paraphrasing the following quotation: "[to pass] from the need of justifying towards the idea of validating within a geometrical system" (Mariotti 1997 sec 4.2)

Synthesis 1…

We use in our field of research a rather large number of key words, among which: proof, argumentation, justification, validation… but, for each of them, we have in mind slightly different meanings when taking mathematics as a reference. This is well known in the now classical exercise, which consists of telling what are the functions of proof. For example, let us take De Villiers or Hanna and Janke propositions (respectively (i) verification, explanation, systematization, discovery and communication, and (ii) construction of an empirical theory, exploration of the meaning of a definition or of the consequences of an hypothesis, imbedding a new fact in a new framework and allowing a new insight). I could add my own propositions, as I did at the very beginning of my work in our field, or those of several other colleagues. Indeed, in several cases the differences in understanding proof and its function may be higher than generally expected.

Is a consensus possible? By consensus here I mean at best a common theoretical framework, at least a glossary guarantying minimally shared meanings. The deadlock on the route towards achieving such a programme is our own epistemology of mathematical proof. By epistemology, I mean here the identification of an object and the web of the relations we establish around it with other objects, as well as problems, tasks and other possible activity involving it.

Whether we consider mathematical proof as a universal and exemplary type of proof (1) or being first of an idiosyncratic nature (2), at the core of mathematics (3) or a tool needed by mathematics (4), getting its meaning from applications (4) or being specific to mathematics as an autonomous field (5), makes a big difference. These views witness radically different epistemologies of mathematical proof, they correspond to very different understandings of mathematical proof in a teaching-learning perspective and hence they will determine the choice of very different research programmes, research design and, above all, radically different understanding of what students could produce.

Indeed, we cannot avoid involving in our work our own epistemology of mathematical proof, and beyond, our own epistemology of mathematics. But if we are not aware of the differences among these epistemologies and the implications of these differences on sharing theories and methods, problems and results, these epistemologies will become the essential obstacle to making progress in our field of research. It is in this sense that researchers epistemology could turn into a deadlock very difficult to brake or to bypass.

Indeed there are common points, which may facilitate the search for common grounds. Among them we can notice: (i) acknowledgement that foundation of mathematical rationality, at least from a learning perspective, is built upon and against a kind of "common sense" rationality coined by historical culture, moral and religious adhesions, professional and social practices of a community; (ii) the existence of deep relationships between argumentation and proof the nature of which is the object of a debate or at least must be turned into a problem; (iii) proof should be considered under the light of both theory and practice; (iv) acknowledgement that mathematics as a content raises specific difficulties either to be bypassed or on the contrary to be built in the emergence of a meaning of mathematical proof; (v) the teacher plays a key role either as a contingent distracter or a required facilitator.

Among all these aspects, surprisingly, one does not show: the relation between proof and language, proving and writing a proof. Before concluding this talk I would like to have a glance at what could be the state of the domain from this perspective.

6

Deductive reasoning holds two characteristics, which oppose it to argumentation. First, it is based on the *operational value of statements* and not on their epistemic value (the belief which may be attached to them). Second, the development of a deductive reasoning relies the possibility of *chaining the elementary deductive steps*, whereas argumentation relies on the reinterpretation or the accumulation of arguments from different points of view. (following Duval 1991, esp. p. 240-241)

Before going ahead, let me start this section by mentioning a very basic remark by Pierre Amiet in his preface of a book on the history of writing. Amiet (1982) emphasises that writing was drastically different from signs, even complex signs, since the aim of writing was not only to reify thoughts, but discourse itself with all its nuances captured in the norm of each language[8]. Indeed, as one can easily realise, in mathematics we do not write as we speak. But if I understand well that mathematicians produce *mathematical texts*, it is not clear for me that they produce discourses—with all the reserve that a non-specialist must take in claiming so. Then, let us proceed with Duval position.

In order to understand Duval claim about mathematical proof, it is necessary to have in mind his definition of “semiotic register” which provides both the theoretical and the methodological framework in which he considers mathematical proof:

> A *semiotic register* …
> … holds traces which can be recognised as the representation of something;
> … provides rules of transformation to produce new representations which could serve to create new knowledge;
> … provides rules for conversion towards an other system of representation to explicit other significations;
> … provides rules of conformity in order to allow the construction of units of a higher level.

This characterisation allows a study of the functional role of writing in constructing a mathematical proof. This does not mean that mathematics is reduced to language, but that the specific character of writing in mathematics has consequences on understanding it and beyond, on understanding mathematical proof. Aside the structural properties, the *ternary step* (hypothesis, theorem, conclusion) and *recycling statements* in order to link two steps (the conclusion of a step becomes the input of an other one), this approach has been used to shape some of the differences between mathematical proof and argumentation. This shaping especially emphasises two features of deductive reasoning: on the one hand theperipheral place of the epistemic value of statements (whether you more or less trust them)—recognising that mathematical proof is apodictic by essence, on the other hand a kind of computational character of deductive reasoning. In Duval sense argumentation and proof are then of a radically different nature. We can imagine how this should raise question in our field whilst

[8] My free translation.

other researchers give a central role to "mathematical arguments" and "mathematical argumentation".

Duval approach should be contrasted with the one of David Pimm (1987). This author, quoting Halliday, recalls that a register is "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (ibid.). And Pimm adds: "Registers have to do with the social usage of particular words, ways of talking but also ways of meaning" (ibid.).

Following Halliday—as Pimm does—one can consider that the function of a text cover "ideational aspects" (express content), "interpersonal aspects" (interaction between the author and the reader) and "textual aspects" (coherence and autonomy). So, if the focus of Duval is on the "textual aspects", the focus of Pimm is rather on the "interpersonal aspects". The latter is especially evident when, referring to Stubbs, Pimm remarks that "the potential use [by teachers] of registers as territorial and status marker rather than as essential to the accurate or concise expression of mathematical ideas" (ibid. p.109)

So, even considering the issue addressed in this talk from the very special point of view of language, one can discover an important discrepancy between the possible underlying epistemology of proof. I will shape this contrast in the following section.

7

> "The conventions of mathematical writing are neither necessary nor natural consequences of the nature of the subject matter; they are rather 'the product of current relations of power and discourse practices'(Clark and Invanik 1997 p.14) within the community" (Burton and Morgan 2000 p.450)

This understanding of mathematical writing drives Burton and Morgan analysis of a corpus of 53 articles written by 70 mathematicians, in which these researchers trace the presence of the author, the expression of authority (negative or positive), the identification of a territory and of a knowledge domain. Finally, the results presented are essentially convergent with Clark and Invanik statement: "Writing, for both students and researchers, is not just about communicating mathematical subject matter. It is also about communicating with individual readers, including powerful gatekeepers such as examiners, reviewers and editors. The writer needs to know how to write in ways that are likely to convince such readers that he or she has the authority to write on this topic, that the subject matter is important enough to be interesting, and that paying attention to what is being said is worthwhile" (ibid. p.451).

Morgan(1998) carried out an extensive research on students' writings in the context of the UK GCSE examination, which allows us to realize the consequences of this view on research on teaching and learning mathematics. For the GCSE examination, students have to write "substantial reports of their work on mathematical investigation". But contrarily to Fawcett emphasis on the benefit from formatting reasoning, the idea there is to privilege a kind of direct access to the underlying understanding of the content and to the richness of the process in a kind of naturalistic manner. The aim of a text in mathematics is viewed as a mean to act upon readers, to persuade them. It should be analysed first from the point of view of argumentation (ibid. p.10). A basic hypothesis is that: "the written (or oral) text is assumed to convey the intention of the author, without distortion or alteration, into the mind of the reader" (ibid. p.196). But again, like in Fawcett case, since students' writings are assessed in the context of an examination, the original intention cannot avoid being distorted by its didactification. Morgan explains and demonstrates well this phenomena which could be seen as the consequence of a double bind introduced by the didactical contract:

- Emphasis on the sincerity of the expression of the student, but what is in the end evaluated is not the product but its author.
- Emphasis on the research process, but the evaluation eventually privileges the content demonstrated.

The literature supporting students and families efforts, and which so to say implements the didactification of students writings, is rather explicit in this sense: "Examine the mathematical content of your work. If the mathematics is merely adding up, then do not expect to gain more than a grade that reflects that you can add up!…" (Morgan quotation, ibid. p.64); or: "You must not be tempted to generalise if it is beyond your ability. It is very easy to spot someone who has tried to generalise without understanding what is involved." (ibid.)

Synthesis 2…

Proof and language are tightly related, especially in mathematics[9]. It is then not surprising to find in research privileging language the same type of discrepancy as the one we find in research addressing directly proof as such. The Halliday's dimensions of a text (ideational, interpersonal, textual) cannot be separated the one from the other in the actual text, and the same is likely to apply when taking the researchers perspectives. But instead, research frameworks make clear and apparently exclusive choices, either focussing on textual aspects (**6**) or interpersonal aspects (**7**). This has a consequence on the results drawn from these researches indeed and the related statements about teaching they suggest. But, this has also consequences on the possibility we have to take benefit from them, and to capitalize their outcomes either to make progress or to make sense of new problems.

Conclusion

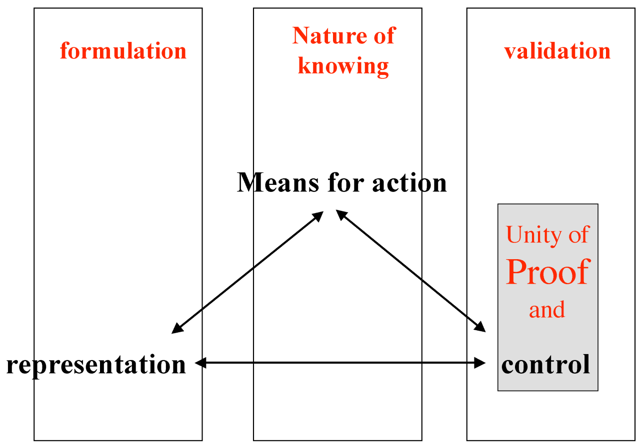
Arsac (1988) asked us the question: "Is it possible to teach mathematical proof?" The answer is nowadays clearly positive. But, it is not clear that we all understand in the same way this question and its possible answer. It is on the contrary clear to me that research speaks in a very confusing way about this topic. Some years ago, ICMI asked us a key question "what is research in mathematics education, and what are its results?" I remember not feeling very comfortable with this question and with what we could tell about it. The case of proof, as I sketch it in this talk is a good illustration of the reasons that made me critical at that time with the way we addressed the issue raised. To be efficient and fruitful, research in mathematics education should pay attention to its coherence across all the specific research projects constituting it. Taking the issue at an international level allows evidencing the problem in an

[9] One may oppose to this statement the case of "proofs without words"; I will not address it here—I did it for the 1991 summer school in *didactique*—but I can ensure that considering this issue does not change drastically the claim insofar as mathematics is concerned (and not just its popularization).

easier way, but I am deeply convinced that it would have been the same considering the issue at a more provincial level.

What emerge from the quick overview I present here, is the role played by the researcher epistemology in his or her choice for a *problématique*, and for his or her choice of a theoretical background and its related methodology. How is it possible to go beyond a mere report of the differences? How is it possible to organise the study of the relationships between "truth" and "validity" within a society, a culture, and the constitution of a didactical *problématique* of mathematical proof—that is, a *problématique* of mathematical proof from the point of view of teaching and learning. How does the rationality of the researcher interfere with or support the research in which he or she is involved? Which role does his or her view play about the acceptable criteria to decide, choose or judge within a mathematical activity taken from a learning perspective? How does that relate to proving?

From a recent reading of Jürgen Habermas, I keep the suggestion that rationality has roots of three different nature: the predicative structure of knowing (or of knowledge at an institutional level), the teleological structure of action and the communication structure of discourse. All these roots are tightly related the one to the other, in other words one may not be able to take one into account without considering the two others. I saw myself a difficulty of this nature when I claimed that validation, communication and nature of knowing cannot be separated in our attempt at understanding what is proving all about. I summarised it in a table as the following:



This schema elicits the relations I proposed in an earlier publication[10] in which I tried to understand the complexity of students understanding of what proving could mean. I was not aware at that time that "control" is a key concept in constructing the unity of problem solving and proving that Boero has extensively explored—the recent Pedemonte's work clarifies this issue and even offers a possible way to solve the problem raised here (Pedemonte 2002).

Would it be possible, recognising the systemic organisation of the relations between representation (the communication level), knowing (the epistemic[11] level) and control (the validation level), to connect our research outcomes and beyond them, our *problématique* and

[10] Balacheff 1987, p.160.
[11] I take here *epistemic* in Piaget sense, that is: "the carrier of knowledge" (in either a social or an individual sense as Furth emphasises).Piaget and Knowledge

theoretical framework. My belief is that it is certainly the case, provided that each of us try to locate his or her own approach among the possible ones, and make the effort to propose an understanding of his or her results from a different perspective. Indeed, I prevent myself from engaging in this exercise now, leaving that for the near future. Especially, I could propose possible links among the different approaches under the light of the theory of didactical situations (especially taking the didactical contract, the problem of devolution and the different type of situations as a key to understand these links). Instead, I would like to propose to start, here in Taipei, to build up a real community of research in a cooperative way.

Could we engage ourselves in the writing of *a Taipei manifesto for research on teaching and learning mathematical proof*. This manifesto could be a kind of recommendation for better practices, which could be offered to our students or colleagues wanting to enter the field. Could we as a community…

- Look for a common lexicon and fix up common definitions possibly acknowledging differences related to our different languages, culture, institutions
- Elicit the different *problématique* and their possible contrast and relationships
- Elicit the theoretical commonalities and divergences, and possibly turn them into questions
- Comment on the different methodologies, their benefit and possible limits
- Acknowledge accepted results or turn objections and differences into research problems
- Stimulate duplication of pieces of work

The new perspective for proving that we are invited to discuss during this conference, could be understood as *the future of research on teaching and learning mathematical proof*. It this is the case, the future of research in the coming decade should be passing from childhood to maturity. Seeking for a rational organisation of our work at an international level, beyond our idiosyncratic views or possible tendency to accept ready made ideas. The task is particularly difficult but not out of reach. It does not mean reaching a flat consensus but an acknowledged awareness of what links and separate our work. In the end, I have every expectation that the benefit of this effort will not be only for research, but significantly for teaching and learning in everyday classes insofar as it will finally be possible for teachers and mathematics educators to make sense of what we publish and declare here and there.

A final note: *this text is more than a draft, but still a first version to be published in the proceedings of the Taipei International Conference on "Mathematics: Understanding Proving and Proving to Understand". Comments, suggestions and questions are welcome.*

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